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Free convection from a flat plate

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A numerical solution is presented for the development of free convection from a semi-infinite vertical flat plate which is uniformly heated up to a length l from the base and insulated for the rest of its length. At great heights above the heated part of the plate, the velocity and temperature distributions behave as if the heat were put in as a line source of heat at the base of the plate. Matching of the solutions for the heated and the insulated parts of the plate, by keeping the fluxes of heat and momentum continuous, determines the position of the effective origin of the similarity solution for the insulated plate in terms of the length, l, of the heated part of the plate. Graphs of the dimensionless velocity, temperature, heat flux and axial length parameters are given for different values of the Prandtl number.

1. Introduction

Fluid motions brought about by the action of body forces (e.g. gravitational, centrifugal, Coriolis or electromagnetic forces) are usually termed as free convection flows as opposed to the forced convection flows in which the fluid flow is maintained mechanically (e.g. by a pressure drop or an agitator). Problems in free convection have mainly been solved by two methods: (*a*) a similarity solution of the partial differential equations governing the flow is sought, which reduces to ordinary differential equations in the similarity variable; (*b*) boundary-layer equations are satisfied on the average by replacing them by their integrals across the boundary layer. Behaviour of the equations at the surface is also utilised in the solution.

Body forces are in equilibrium with the hydrostatic pressure when there is no temperature difference between the plate and its environment and no flow will develop in the steady state. If the plate is heated the resultant rise in temperature in the fluid produces a defect of body force because of decreased density. The fluid nearer to the plate is therefore subject to a buoyancy force which is balanced by the viscous forces and the inertia of the moving fluid.

The earliest analysis and experiments on free convection flow under gravity about an isothermal flat plate was done by Schmidt & Beckman (1930). Ostrach (1953) reformulated the problem considering a viscous and conducting fluid and developing asymptotic forms of the basic equations. For non-isothermal surfaces Sparrow & Gregg (1958) have given a solution of the boundary-layer equations of free convection from a vertical flat plate for two families of surface temperature for different values of the Prandtl number.

Takhar (1967) obtained a numerical solution for the development of laminar free convection near a vertical insulated flat plate which is heated at the base by a line source of heat. The condition of insulation effectively reduces the problem to that of free convection near a flat plate whose temperature varied as $(x)^{-\frac{3}{5}}$ where x is the distance from the leading edge. A Pohlhausen type of solution suggests the form of the similarity parameters which are used in solving the boundary-layer equations numerically for different values of the Prandtl number. Takhar also studied the problem of free convection produced by heating a length, l, of the semi-infinite vertical flat plate and insulating the rest of its length. At a great height above the heated part the velocity and temperature distributions behave as if the heat were put in as a line source of heat near the base of the plate. A Pohlhausen solution given for this case determines the effective origin of the similarity solution in terms of the length, l, of the uniformly heated part of the plate.

The present paper provides a numerical solution to the problem of free convection from a vertical flat plate which is partly uniformly heated and partly insulated. This is done by matching Ostrach's (1953) numerical solution for a uniformly heated, vertical flat plate and Takhar's (1967) solution for an insulated flat plate (with displaced origin) by keeping the fluxes of heat and momentum continuous. Solutions are presented for parametric values of the Prandtl number for the dimensionless velocity, temperature, heat flux and axial length parameters in the form of graphs.

2. Governing equations

Consider a two-dimensional frame of reference in which the origin is at the lower edge of the plate, x-axis along the plate and y-axis normal to the plate with the associated velocity components u and v in these directions. Neglect (i) any variations in the kinematic viscosity ν and the thermometric conductivity κ, (ii) viscous dissipation and the work done against the gravity field. By assuming the Boussinesq approximation, whereby we neglect the variations in density except in the buoyancy term, and absorbing the gravity in the hydrostatic pressure, the governing equations expressing the conservation of mass, momentum and heat energy for a steady laminar flow in the boundary layer formed along the flat plate reduce to

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = g\beta(T-T_\infty)+\nu\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

$$u\frac{\partial T}{\partial x}+u\frac{\partial T}{\partial y} = \kappa\frac{\partial^2 T}{\partial y^2}, \tag{3}$$

with the associated boundary conditions

$$\left.\begin{aligned} u &= 0 \\ v &= 0 \\ T &= T_w \quad \text{for} \quad x < l \\ \partial T/\partial y &= 0 \quad \text{for} \quad x > l \end{aligned}\right\} \text{at the wall;} \quad \left.\begin{aligned} u &= 0 \\ T &= T_\infty \\ \partial u/\partial y &= 0 \\ \partial T/\partial y &= 0 \end{aligned}\right\} \text{at infinity;} \tag{4}$$

where g is the acceleration due to gravity, β the coefficient of cubical expansion, T the temperature of the air, T_∞ the temperature at infinity and T_w the temperature at the wall. The continuity equation (1) defines a stream function ψ given by

$$u = \partial\psi/\partial y; \quad v = -\partial\psi/\partial x. \tag{5}$$

3. Uniformly heated plate

For the uniformly heated plate Ostrach (1953) defined similarity variables of the form

$$\eta = \left[\frac{Gr}{4}\right]^{\frac{1}{4}} \frac{y}{x}, \tag{6}$$

$$Gr = \frac{g\beta(T_w - T_\infty)}{\nu^2} x^3, \tag{7}$$

$$H(\eta) = \frac{T - T_\infty}{T_w - T_\infty} \tag{8}$$

and

$$\psi = 4\nu[\tfrac{1}{4}Gr]^{\frac{1}{4}} F(\eta). \tag{9}$$

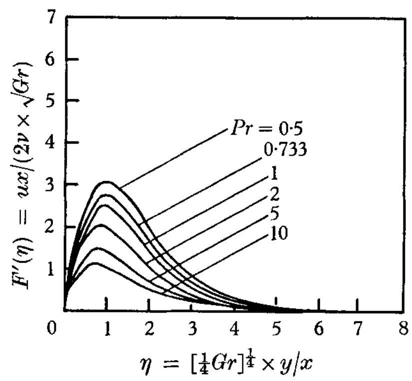

FIGURE 1. Dimensionless velocity distributions for various Prandtl numbers for the uniformly heated plate.

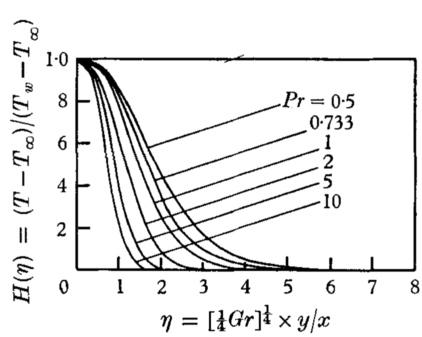

FIGURE 2. Dimensionless temperature distributions for various Prandtl numbers for the uniformly heated plate.

Substitution of these similarity parameters in the governing equations (1), (2), (3) leads to the ordinary differential equations in the similarity variable η;

$$F''' + 3FF'' - 2F'^2 + H = 0, \tag{10}$$

$$H'' + 3Pr\,FH' = 0, \tag{11}$$

where

$$Pr = \nu/\kappa. \tag{12}$$

The boundary conditions now become

$$\left.\begin{aligned} F'(0) &= 0 = F(0), \\ F'(\infty) &= 0 = H(\infty). \end{aligned}\right\} \tag{13}$$

These equations are solved numerically for different values of the Prandtl number. Results for the development of the non-dimensional velocity and temperature profiles are given by figures 1 and 2.

The characteristic non-dimensional number of the problem may be defined through the constant heat flux integral

$$\int_0^\infty u(T-T_\infty)\,dy$$

as
$$K = \left[\frac{g\beta x^3}{\nu^2\kappa}\int_0^\infty u(T-T_\infty)\,dy\right] = 16Pr\left[\frac{Gr}{4}\right]^{\frac{5}{4}}\int_0^\infty F'H\,d\eta. \tag{14}$$

Results for the development of the non-dimensional heat-flux parameter for different values of the Prandtl number are given in figure 3.

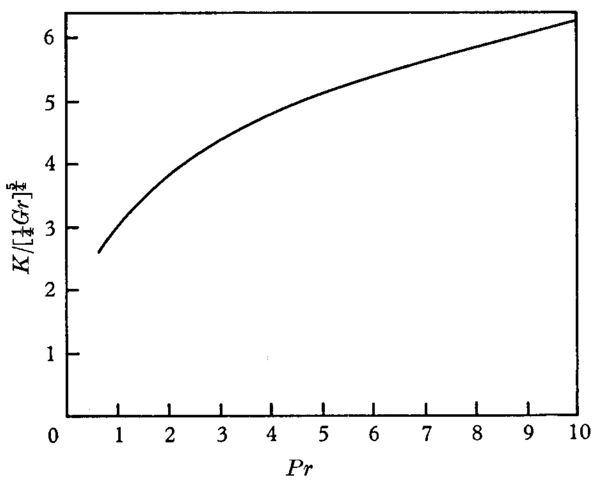

FIGURE 3. Dimensionless heat flux parameter as a function of the Prandtl number for the uniformly heated plate.

4. Insulated plate

For the insulated plate a similarity solution was obtained by Takhar (1967) by using similarity variables of the form (if the origin is now assumed displaced by a distance, a, from the leading edge)

$$\eta^* = \left[\frac{Gr^*}{4}\right]^{\frac{1}{4}}\frac{y}{x-a}, \tag{15}$$

$$Gr^* = \frac{g\beta(T_w-T_\infty)(x-a)^3}{\nu^2}, \tag{16}$$

$$\theta(\eta^*) = \frac{T-T_\infty}{T_w-T_\infty}, \tag{17}$$

$$\psi^* = 4\nu[\tfrac{1}{4}Gr^*]^{\frac{1}{4}}f(\eta^*). \tag{18}$$

Substitution of these similarity variables into the governing equations (1), (2), (3) leads to the ordinary differential equations in the similarity variable η^*;

$$f''' - \tfrac{4}{5}f'^2 + \tfrac{12}{5}ff'' + \theta = 0, \tag{19}$$

$$\theta'' = (-\tfrac{12}{5}Pr)(f'\theta + f\theta'). \tag{20}$$

The last equation (20) integrates to

$$\theta' = (-\tfrac{12}{5}Pr)f\theta, \tag{21}$$

since $\theta(\infty) = 0$. This shows that $\theta'(0) = 0$, which is the condition that there is no heat transfer.

The boundary conditions on the problems now reduce to

$$\left.\begin{matrix} f = 0 \\ f' = 0 \\ \theta = 1 \end{matrix}\right\} \text{ at } \eta^* = 0; \quad \left.\begin{matrix} f' = 0 \\ \theta' = 0 \end{matrix}\right\} \text{ at } \eta^* = \infty. \tag{22}$$

These ordinary differential equations have been numerically solved for different values of the Prandtl numbers. Results for the development of the non-dimensional velocity and temperature profiles are given in figures 4 and 5.

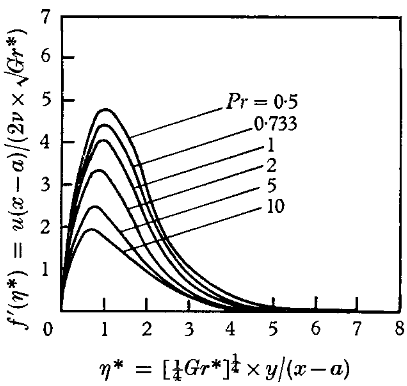

FIGURE 4. Dimensionless velocity distributions for various Prandtl numbers for the insulated plate.

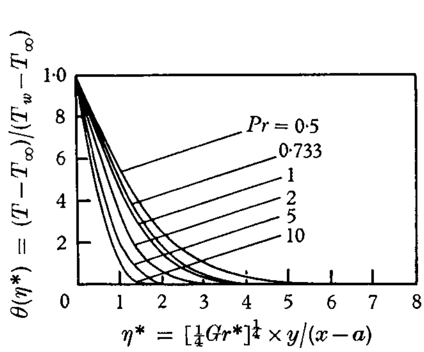

FIGURE 5. Dimensionless temperature distributions for various Prandtl numbers for the insulated plate.

The characteristic non-dimensional number of the problem may be defined through the constant heat flux integral,

$$\int_0^\infty u(T-T_\infty)\,dy$$

as

$$K^* = \left[\frac{g\beta(x-a)^3}{\nu^2\kappa}\int_0^\infty u(T-T_\infty)\,dy\right] = 16Pr[\tfrac{1}{4}Gr^*]^{\frac{5}{4}}\int_0^\infty f'\theta\,d\eta^*. \tag{23}$$

Results for the development of the non-dimensional heat flux parameter for different values of the Prandtl number are given in figure (6).

5. Matching

The constants a and l which respectively represent the displacement of the origin in Takhar's solution and the length of the heated part of the plate can be found by matching the two solutions at $x = l$ where

$$\int_0^\infty u^2\,dy \quad \text{and} \quad \int_0^\infty u(T-T_\infty)\,dy$$

are both continuous. From Ostrach's solution

$$\int_0^\infty u^2 dy = 16[\tfrac{1}{4}Gr]^{\frac{3}{4}} \frac{\nu^2}{x} \int_0^\infty F'^2 d\eta, \tag{24}$$

$$\int_0^\infty u(T-T_\infty) dy = 4\nu[\tfrac{1}{4}Gr]^{\frac{1}{4}} (T_w - T_\infty) \int_0^\infty F'H\, d\eta. \tag{25}$$

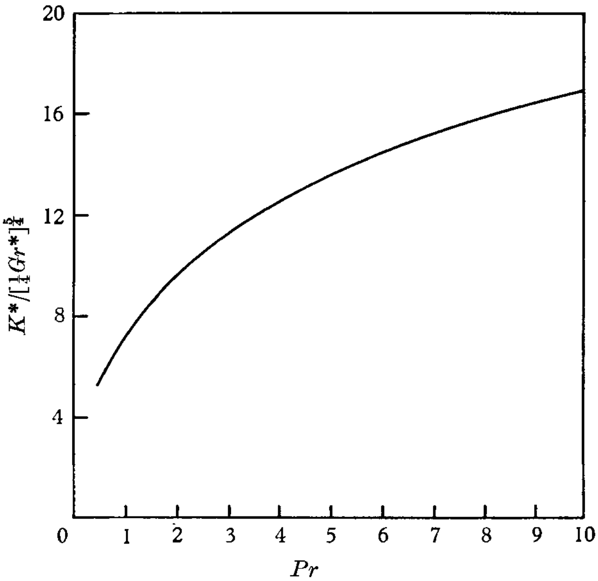

FIGURE 6. Dimensionless heat flux parameter as a function of the Prandtl number for the insulated plate.

From Takhar's solution (with displaced origin)

$$\int_0^\infty u^2 dy = 16[\tfrac{1}{4}Gr^*]^{\frac{3}{4}} \frac{\nu^2}{x-a} \int_0^\infty f'^2 d\eta^*, \tag{26}$$

$$\int_0^\infty u(T-T_\infty) dy = 4\nu[\tfrac{1}{4}Gr^*]^{\frac{1}{4}} (T_w - T_\infty) \int_0^\infty f'\theta\, d\eta^*. \tag{27}$$

Equating (24) and (26) at $x = l$ we get

$$16[\tfrac{1}{4}Gr]^{\frac{3}{4}}_{x=l} \frac{\nu^2}{l} \int_0^\infty F'^2 d\eta = 16[\tfrac{1}{4}Gr^*]^{\frac{3}{4}}_{x=l} \frac{\nu^2}{x-l} \int_0^\infty f'^2 d\eta^*.$$

This leads to

$$\frac{\int_0^\infty F'^2 d\eta}{\int_0^\infty f'^2 d\eta^*} = \left(\frac{Gr^*}{Gr}\right)^{\frac{3}{4}} \frac{l}{l-a} = \left(1 - \frac{a}{l}\right)^{\frac{4}{5}}. \tag{28}$$

Equating (25) and (27) at $x = l$ we get

$$4\nu(\tfrac{1}{4}Gr)^{\frac{1}{4}}_{x=l} (T_w - T_\infty) \int_0^\infty F'H\, d\eta = 4\nu(\tfrac{1}{4}Gr^*)^{\frac{1}{4}}_{x=l} (T_w - T_\infty) \int_0^\infty f'\theta\, d\eta^*.$$

This leads to

$$\frac{\int_0^\infty F'H\,d\eta}{\int_0^\infty f'\theta\,d\eta^*} = \left[\frac{Gr^*}{Gr}\right]_{x=l}^{\frac{1}{4}} = \left(1-\frac{a}{l}\right)^{\frac{3}{4}}. \tag{29}$$

From (28) and (29) we get,

$$\left(1-\frac{a}{l}\right)^{\frac{1}{2}} = \frac{\int_0^\infty F'^2\,d\eta \Big/ \int_0^\infty f'^2\,d\eta^*}{\int_0^\infty F'H\,d\eta \Big/ \int_0^\infty f'\theta\,d\eta^*}.$$

This leads to

$$\left(1-\frac{a}{l}\right) = \left[\frac{\int_1^\infty F'^2\,d\eta \Big/ \int_0^\infty f'^2\,d\eta^*}{\int_0^\infty F'H\,d\eta \Big/ \int_0^\infty f'\theta\,d\eta^*}\right]^2. \tag{30}$$

Results for the non-dimensional length parameter, $(1-a/l)$, for various values of the Prandtl number are given in figure 7.

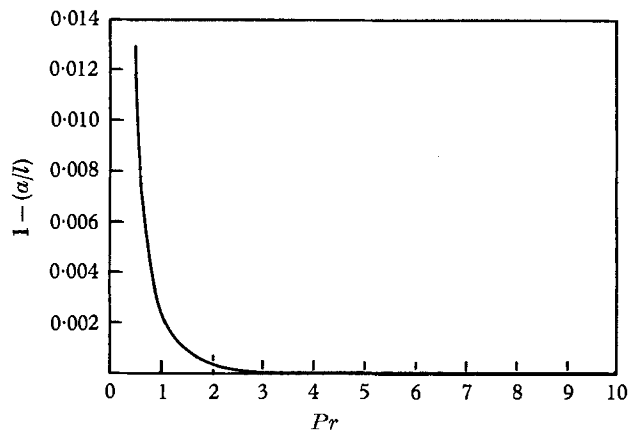

FIGURE 7. Dimensionless axial length parameter $(1-a/l)$) as a function of the Prandtl number.

6. Conclusions

Distributions of velocity, temperature and heat flux are shown in figures 1–3 for the case of a uniformly heated vertical flat plate and in figures 4–6 for the insulated vertical flat plate. For $Pr < 1$ the buoyancy forces are dominant over the viscous forces and relatively higher velocities are observed for decreasing values of the Prandtl number. However, for $Pr > 1$ when viscous forces are dominant over the buoyancy forces, relatively lower velocities are observed for increasing values of the Prandtl number. Also, for the insulated flat plate higher velocities are observed than the uniformly heated plate for the corresponding Prandtl numbers. The value of the non-dimensional temperature parameters H and θ is equal to 1 at the wall and it gradually tapers off to zero with increasing values of η and η^* respectively, but only more slowly (i) for the insulated than the uniformly heated plate and (ii) for the lower (than 1) values of the Prandtl number.

Velocity accelerates more rapidly and the temperature decays more slowly (for $Pr < 1$) for the insulated flat plate than in the case of the uniformly heated plate. This is also in accord with the non-dimensional parameter $K^*/(\frac{1}{4}Gr^*)^{\frac{1}{4}}$ for the insulated plate developing more rapidly than the corresponding parameter

Pr	$(1-a/l)$	$K/(\frac{1}{4}Gr)^{\frac{1}{4}}$	$K^*/(\frac{1}{4}Gr^*)^{\frac{1}{5}}$
0·5	0·0125	2·3520	5·2144
0·733	0·0055	2·7072	6·3828
1	0·0023	3·0248	7·1540
2	0·0004	3·8216	2·5800
5	0	5·0880	13·5208
10	0	6·2368	17·0784

TABLE 1. Values of the non-dimensional ratios for various Prandtl numbers.

$K/(\frac{1}{4}Gr)^{\frac{1}{4}}$ for the uniformly heated plate. Matching of these two solutions gives an effective displacement of the similarity solution in the vertical direction. For large Prandtl numbers the effective origin is close to the point $x = l$ but if the Prandtl number is small enough the effective origin may be below the leading edge of the plate. Figure 7 shows the distribution of the dimensionless length parameter $(1-a/l)$ as a function of the Prandtl number. The value of this parameter is very nearly equal to zero for $Pr = 0{\cdot}5$ and tends to the limit zero for $Pr \geqslant 5$ indicating that the effective origin is nearly all the way up the heated part of the plate.

7. Numerical integration

As no analytical solutions can be easily found for the sets of equations (10) and (11) and (19) and (20) these were solved numerically under the boundary conditions (13) and (22) respectively by using a marching solution from $\eta = 0$ and $\eta^* = 0$. Before we can use the Kutta–Merson techniques of solution of simultaneous ordinary differential equations we have to know $F''(0)$; $H'(0)$ and $f''(0)$ in the two cases respectively. This was achieved by using the 'Haselgrove 2-point Boundary Value Program' which is stored on a magnetic tape in the Atlas Computer at the Manchester University. The Haselgrove Program integrates the given system of differential equations from the origin outwards and from the infinity inwards. These solutions are matched at an intermediate point which must be guessed. The fitting is done by adjusting the guesses for the unknown boundary values so that the differences in the functions obtained by the forward and backward integrations are as small as possible. This process of guessing and fitting was repeated for different values of the Prandtl number to get the starting values of the functions $H''(0)$, $H'(0)$ and $f''(0)$. After this the Kutta–Merson Program is used to integrate the given systems of differential equations outward from $\eta = 0$ and $\eta^* = 0$ respectively.

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